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*The General Theory of Linear q -Difference Equations.**

BY R. D. CARMICHAEL.

Introduction.

By means of a transformation of the form $z = (m_1x + m_2)/(\mu_1x + \mu_2)$, the system of functional equations

$$H_i\left(\frac{az + b}{cz + d}\right) = \sum_{j=1}^n \lambda_{ij}(z) H_j(z) \quad (i = 1, \dots, n),$$

in the n unknown functions $H_1(z), \dots, H_n(z)$, may be transformed into the system of difference equations

$$G_i(x + 1) = \sum_{j=1}^n \bar{\lambda}_{ij}(x) G_j(x) \quad (i = 1, \dots, n),$$

or into the system of q -difference equations

$$G_i(qx) = \sum_{j=1}^n \bar{\lambda}_{ij}(x) G_j(x) \quad (i = 1, \dots, n),$$

according as the substitution $z' = (az + b)/(cz + d)$ has one or two double points. To do this it is necessary to choose the transforming substitution so that in the first case the single double point is carried to infinity, and in the second case the two double points are carried to zero and to infinity respectively.

The essential general properties of the solutions of linear difference equations are known.†

The present paper is devoted to an investigation of the existence and properties of solutions of linear q -difference equations.

In § 1, for the case when $|q| \neq 1$, I prove the existence of two fundamental systems of solutions, one of simple character at infinity and the other of simple

* Read before the American Mathematical Society (Chicago), April 28, 1911.

† See the papers by Carmichael and by Birkhoff in *Transactions of the American Mathematical Society*, Vol. XII. References to the previous literature of difference equations will be found in these papers.

character at zero. These two systems of solutions are analogous to the two systems of solutions of difference equations whose existence I pointed out at the close of my difference-equation paper (*loc. cit.*, pp. 133–134).

In § 2 an investigation of the relations between these two fundamental systems of solutions leads to a theory analogous to the interesting Birkhoff characterization of the solutions of a system of difference equations (see §§ 5 and 7 of the paper already referred to).

In § 3 I consider the exceptional case when $|q| = 1$. A method is given for obtaining fundamental systems of solutions in explicit finite form.

Rev. F. H. Jackson* has given a treatment of some questions connected with q -difference equations. Reference should also be made to some of the general theorems on functional equations due to A. Grévy† and L. Leau,‡ from which some of the results of the present paper can be deduced.

§ 1. *Existence of two Fundamental Systems of Solutions when $|q| \neq 1$.*

Let
$$G_i(qx) = q^\alpha x^\alpha \sum_{j=1}^n A_{ij}(x) G_j(x) = q^\beta x^\beta \sum_{j=1}^n B_{ij}(x) G_j(x) \quad (i = 1, \dots, n) \quad (1)$$

be a system of n first-order linear homogeneous q -difference equations involving the n unknown functions $G_1(x), \dots, G_n(x)$ of the complex variable x , the known quantities entering into the equation being defined as follows:

1. α, β, q are constants and $|q| \neq 1$.
2. The functions $A_{ij}(x)$ and $B_{ij}(x)$ are single-valued and

$$\begin{aligned} A_{ij}(x) &= A_{ij} + A'_{ij}x^{-1} + A''_{ij}x^{-2} + \dots & (i, j = 1, \dots, n) & \quad |x| \geq R, \\ B_{ij}(x) &= B_{ij} + B'_{ij}x + B''_{ij}x^2 + \dots & (i, j = 1, \dots, n) & \quad |x| \leq r. \end{aligned}$$

3. The constants A_{ij} and the constants B_{ij} are such that the roots A_1, \dots, A_n and B_1, \dots, B_n respectively of the characteristic equations at infinity and at zero,

$$|A_{ij} - \delta_{ij}\rho| = 0, \quad |B_{ij} - \delta_{ij}\rho| = 0, \quad \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

are finite and different from zero and verify the relations

$$q^\gamma A_i - A_j \neq 0, \quad q^\gamma B_i - B_j \neq 0 \quad (i, j = 1, \dots, n; i \neq j),$$

γ being any positive or negative integer or zero.

* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII, pp. 305–314.

† Paris thesis, 1894.

‡ Paris thesis, 1897.

When the roots of a characteristic equation satisfy the given relations, we shall say that they are of simple character.

By means of the transformations

$$G_i(x) = e^{\frac{\alpha}{x} \log q (\eta^2 + \eta)} \bar{G}_i(x) = e^{\frac{\beta}{x} \log q (\eta^2 + \eta)} \bar{G}'_i(x) \quad (i = 1, \dots, n), \quad (2)$$

where $\eta = \log x / \log q$, system (1) may be replaced by two simpler systems, one of which has $\alpha = 0$ and the other of which has $\beta = 0$. Making these substitutions and reducing, we have

$$\bar{G}_i(qx) = \sum_{j=1}^n A_{ij}(x) \bar{G}_j(x) \quad (i = 1, \dots, n), \quad (3)$$

$$\bar{G}'_i(qx) = \sum_{j=1}^n B_{ij}(x) \bar{G}'_j(x) \quad (i = 1, \dots, n). \quad (4)$$

For the sake of simplicity in formulæ it is convenient to reduce (3) and (4) to normal forms valid in the neighborhood of infinity and of zero respectively. We begin with system (3).

Let us put

$$\bar{G}_i(x) = \sum_{j=1}^n \alpha_{ij} F_j(x) \quad (i = 1, \dots, n), \quad (5)$$

where α_{ij} , $i, j = 1, \dots, n$, are constants whose determinant $|\alpha_{ij}|$ is different from zero. Making this substitution in (3) and solving for $F_i(qx)$, $i = 1, \dots, n$, one may write the result in the form

$$F_i(qx) = \sum_{j=1}^n \bar{A}_{ij}(x) F_j(x) \quad (i = 1, \dots, n),$$

where the functions $\bar{A}_{ij}(x)$, being linear combinations of the functions $A_{ij}(x)$ with constant coefficients, may be expanded in powers of x^{-1} for $|x| \geq R$. From the well-known theory of linear substitutions* and the fact that the roots of the characteristic equation at infinity are of simple character, it follows that a proper determination of the constants α_{ij} will reduce the constant term in $\bar{A}_{ij}(x)$ to zero when $i \neq j$. The preceding system of equations may then be written in the form

$$F_i(qx) = A_i F_i(x) + \sum_{j=1}^n (\alpha_{ij} x^{-1} + \alpha'_{ij} x^{-2} + \dots) F_j(x) \quad (6)$$

$$(i = 1, \dots, n) \quad |x| \geq R.$$

*Compare my difference-equation paper, *loc. cit.*, pp. 126-127, where for a similar transformation the reduction is carried out in detail.

In order to remove the term in x^{-1} from each of the non-diagonal coefficient functions of the second members, we may employ the transformation

$$F_i(x) = \bar{F}_i(x) + \sum_{j=1}^n \frac{\beta_{ij}}{x} \bar{F}_j(x), \quad \beta_{ii} = 0, \quad (i = 1, \dots, n), \quad (7)$$

where the constants β_{ij} , $i \neq j$, are to be determined. Clearly the determinant of the transformation is not identically zero. If we substitute the above values of $F_i(x)$ in the preceding system of equations, and in the result write only those terms which do not involve x^{-2} , x^{-3} , \dots , we have, when $|x| \geq R$,

$$\begin{aligned} & \bar{F}_i(qx) + \sum_{j=1}^n \frac{\beta_{ij}}{qx} \bar{F}_j(qx) \\ &= A_i \bar{F}_i(x) + A_i \sum_{j=1}^n \frac{\beta_{ij}}{x} \bar{F}_j(x) + \sum_{j=1}^n \left\{ \left(\frac{\alpha_{ij}}{x} + \dots \right) \left[\bar{F}_j(x) + \sum_{k=1}^n \frac{\beta_{jk}}{x} \bar{F}_k(x) \right] \right\} \\ &= \left(A_i + \frac{\alpha_{ii}}{x} + \dots \right) \bar{F}_i(x) + \sum_{j=1}^n \left(\frac{A_i \beta_{ij} + \alpha_{ij}}{x} + \dots \right) \bar{F}_j(x) \quad (i=1, \dots, n), \end{aligned}$$

where the accent in Σ' denotes that in the summation the term for which $j = i$ is to be omitted.

In order to solve these equations for $\bar{F}_i(qx)$, $i = 1, \dots, n$, in terms of $\bar{F}_i(x)$, $i = 1, \dots, n$, we shall first find the cofactors of the determinant of the system,

$$\Delta \equiv \left| \delta_{ij} + \frac{\beta_{ij}}{qx} \right| = 1 + \text{terms in } x^{-2}, x^{-3}, \dots \quad |x| \geq R.$$

The cofactor Δ_{ij} of the element in the i -th row and the j -th column has evidently no constant term, except when i equals j , when there is a constant term 1. The term in x^{-1} for $i \neq j$ is clearly $-\beta_{ji}/qx$. For, if we cross out the i -th row and the j -th column of Δ , there remain $n - 2$ elements with a constant term 1; and therefore the expansion of this minor can contain only one term in x^{-1} , namely, that term which contains all the $n - 2$ factors 1 and whose remaining factor is not in the same rows or columns as these. Therefore this term is $\pm \beta_{ji}/qx$; and, since an even number of interchanges of rows and columns brings the elements β_{ij}/qx and β_{ji}/qx to the place in the first row second column, and second row first column respectively, the negative sign must be chosen. In the cofactor Δ_{ii} there is no term in x^{-1} ; for every term in the expansion containing

one factor x^{-1} necessarily contains another. Hence the cofactors may be written

$$\begin{aligned}\Delta_{ij} &= -\frac{\beta_{ji}}{qx} + \text{terms in } x^{-2}, x^{-3}, \dots & (i \neq j); \\ \Delta_{ii} &= 1 + \text{terms in } x^{-2}, x^{-3}, \dots\end{aligned}$$

Solving the last system of equations by the aid of these cofactors, we readily obtain

$$\Delta \bar{F}_i(qx) = \left(A_i + \frac{a_{ii}}{x} + \dots \right) \bar{F}_i(x) + \sum_{j=1}^n \left\{ \frac{(qA_i - A_j)\beta_{ij} + qa_{ij} + \dots}{x} \right\} \bar{F}_j(x) \\ (i = 1, \dots, n).$$

Since the roots of the characteristic equation at infinity are of simple character, it follows that it is always possible to determine β_{ij} , $i \neq j$, so that $(qA_i - A_j)\beta_{ij} + qa_{ij} = 0$. If the values of β_{ij} so determined are substituted in the preceding system and each of the equations is then divided by Δ , the result takes the form

$$\bar{F}_i(qx) = \left(A_i + \frac{a_{ii}}{x} \right) + \sum_{j=1}^n \left(\frac{b_{ij}}{x^2} + \frac{b'_{ij}}{x^3} + \dots \right) \bar{F}_j(x) \quad (8) \\ (i = 1, \dots, n) \quad |x| \geq R.$$

It is now easy to see that by means of the transformation

$$\bar{F}_i(x) = \bar{f}_i(x) + \sum_{j=1}^n \frac{\gamma_{ij}}{x^2} \bar{f}_j(x), \quad \gamma_{ii} = 0, \quad (i = 1, \dots, n), \quad (9)$$

system (8) goes over into the form

$$\bar{f}_i(qx) = (A_i + a_{ii}x^{-1} + b_{ii}x^{-2})\bar{f}_i(x) + \sum_{j=1}^n (c_{ij}x^{-3} + \dots)\bar{f}_j(x) \\ (i = 1, \dots, n) \quad |x| \geq R.$$

By repeated use of transformations similar to (7) and (9), the exponent of x in the coefficients of the transformation being increased by unity at each step, it is clear that our system of equations may be reduced to

$$f_i(qx) = (A_i + A'_i x^{-1} + \dots + A_i^{(s)} x^{-s})f_i(x) + \sum_{j=1}^n (d_{ij}x^{-s-1} + \dots)f_j(x) \quad (10) \\ (i = 1, \dots, n) \quad |x| \geq R,$$

where s is any positive integer.

By combining into one all the transformations thus far employed one readily proves that *the change from (1) to (10) may be effected at once by a transformation of the type*

$$G_i(x) = e^{\frac{\alpha}{2} \log q (\eta^2 + \eta)} \sum_{j=1}^n \pi_{ij} \left(\frac{1}{x} \right) f_j(x) \quad (i = 1, \dots, n), \quad (11)$$

where the $\pi_{ij} \left(\frac{1}{x} \right)$ are polynomials in $1/x$, the determinant of the transformation not being identically zero.

In a similar way one may find polynomials $P_{ij}(x)$ in x such that the transformation

$$G_i(x) = e^{\frac{\beta}{2} \log q (\eta^2 + \eta)} \sum_{j=1}^n P_{ij}(x) g_j(x) \quad (i = 1, \dots, n) \quad (12)$$

is non-singular and carries (1) over into

$$g_i(qx) = (B_i + B'_i x + \dots + B_i^{(\sigma)} x^\sigma) g_i(x) + \sum_{j=1}^n (e_{ij} x^{\sigma+1} + \dots) g_j(x) \quad (13)$$

$$(i = 1, \dots, n) \quad |x| \leq r,$$

where σ is any positive integer. It is hardly necessary to remark that the forms of the various preliminary transformations here are what those of the preceding case become when $1/x$ is replaced by x .

If $A = q^\mu$, it is easy to determine constants $c', c'', \dots, c^{(2s)}$ such that

$$f(x) = x^\mu (1 + c'x^{-1} + \dots + c^{(s)}x^{-s})$$

verifies the equation

$$f(qx) = (A + A'x^{-1} + \dots + A^{(s)}x^{-s})f(x) + x^\mu (c^{(s+1)}x^{-s-1} + \dots + c^{(2s)}x^{-2s}).$$

Evidently this may be written

$$f(qx) = \left(A + \dots + A^{(s)}x^{-s} + \frac{c^{(s+1)}x^{-s-1} + \dots + c^{(2s)}x^{-2s}}{1 + c'x^{-1} + \dots + c^{(s)}x^{-s}} \right) f(x) = q^\mu \lambda(x) f(x),$$

where $\lambda(x)$ has the constant term 1 and is analytic when $|x|$ is greater than the greatest absolute value of a zero of $1 + c'x^{-1} + \dots + c^{(s)}x^{-s}$. Consequently, if we write $A_i = q^{\mu_i}$, there exist functions $\lambda_i(x)$ and a constant \bar{R} greater than or equal to R such that system (10) may be written in the form

$$f_i(qx) - q^{\mu_i} \lambda_i(x) f_i(x) = \sum_{j=1}^n \psi_{ij}(x) f_j(x) \quad (i = 1, \dots, n), \quad (14)$$

where each of the $n^2 + n$ functions $\psi_{ij}(x)$, $\lambda_i(x)$ is analytic for $|x| \geq \bar{R}$, the

expansion for $\psi_{ij}(x)$ beginning with a term in x^{-s-1} , and where the functions $\lambda_i(x)$ have each the constant term 1 and are such that for every i the equation

$$f_i^{(1)}(qx) - q^{\mu_i} \lambda_i(x) f_i^{(1)}(x) = 0$$

has a solution in the form

$$f_i^{(1)}(x) = U_i(x) \equiv x^{\mu_i} (1 + c'_i x^{-1} + \dots + c_i^{(s)} x^{-s}).$$

In a similar way it may be shown that if $B_i = q^{m_i}$ there exist functions $l_i(x)$ and a constant \bar{r} different from zero and equal to or less than r such that system (13) may be written in the form

$$g_i(qx) - q^{m_i} l_i(x) g_i(x) = \sum_{j=1}^n \phi_{ij}(x) g_j(x) \quad (i = 1, \dots, n), \quad (15)$$

where each of the $n^2 + n$ functions $\phi_{ij}(x)$, $l_i(x)$ is analytic for $|x| \leq \bar{r}$, the expansion for $\phi_{ij}(x)$ beginning with a term in $x^{\sigma+1}$, and where the functions $l_i(x)$ have each the constant term 1 and are such that for every i the equation

$$g_i^{(1)}(qx) - q^{m_i} l_i(x) g_i^{(1)}(x) = 0$$

has a solution in the form

$$g_i^{(1)}(x) = V_i(x) \equiv x^{m_i} (1 + \bar{c}'_i x + \dots + \bar{c}_i^{(\sigma)} x^{\sigma}).$$

Systems (14) and (15) are the normal forms of (1), the first being valid in the neighborhood of infinity and the second in the neighborhood of zero. To each of these normal forms the method of successive approximation is conveniently applicable. By means of this method we shall now prove the existence of a fundamental system of solutions of the equations in the normal form (14).

Consider the set of systems each of n linear equations, all except the first being non-homogeneous :

$$\left. \begin{aligned} f_i^{(1)}(qx) - q^{\mu_i} \lambda_i(x) f_i^{(1)}(x) &= 0, \\ f_i^{(2)}(qx) - q^{\mu_i} \lambda_i(x) f_i^{(2)}(x) &= \sum_{j=1}^n \psi_{ij}(x) f_j^{(1)}(x), \\ f_i^{(3)}(qx) - q^{\mu_i} \lambda_i(x) f_i^{(3)}(x) &= \sum_{j=1}^n \psi_{ij}(x) f_j^{(2)}(x), \\ &\dots \dots \dots \end{aligned} \right\} \quad (16)$$

We have already found a solution $U_i(x)$ of the first of these systems of equations ; we shall consider here the more general solution

$$u_i(x) = C_i(x) U_i(x) \equiv C_i(x) x^{\mu_i} (1 + c'_i x^{-1} + \dots + c_i^{(s)} x^{-s}) \quad (i = 1, \dots, n), \quad (17)$$

where $C_i(x)$ is any function satisfying the functional equation $C_i(qx) = C_i(x)$.

All the other systems have the general non-homogeneous form

$$h_i(qx) - q^{\mu_i} \lambda_i(x) h_i(x) = \eta_i(x) \quad (i = 1, \dots, n), \quad (18)$$

only a single unknown function $h_i(x)$ entering into the i -th equation of the system $i = 1, \dots, n$. Two formal solutions are readily obtained as follows: Let

$$h_i(x) = U_i(x) \bar{h}_i(x)$$

and substitute in the preceding equation. Dividing the result by

$$U_i(qx) = q^{\mu_i} \lambda_i(x) U_i(x),$$

we have

$$\bar{h}_i(qx) - \bar{h}_i(x) = \frac{\eta_i(x)}{U_i(qx)},$$

a system with the two formal solutions

$$\bar{h}_i(x) = - \sum_{v=0}^{\infty} \frac{\eta_i(q^v x)}{U_i(q^{v+1} x)}, \quad \bar{h}_i(x) = \sum_{v=0}^{\infty} \frac{\eta_i(q^{-v-1} x)}{U_i(q^{-v} x)}.$$

Hence two formal solutions of (18) are

$$h_i(x) = - U_i(x) \sum_{v=0}^{\infty} \frac{\eta_i(q^v x)}{U_i(q^{v+1} x)}, \quad h_i(x) = U_i(x) \sum_{v=0}^{\infty} \frac{\eta_i(q^{-v-1} x)}{U_i(q^{-v} x)}. \quad (19)$$

Either of these may be used to obtain a sequence of formal solutions of (16). It will turn out that one of these sequences will lead to a (convergent) solution and the other to a divergent formal solution of (14). The first leads to a (convergent) solution if $|q| > 1$, the second if $|q| < 1$. As the reasoning is entirely similar in the two cases, it is sufficient to carry it out for one. We shall suppose that $|q| > 1$ and shall therefore use the first formal solution (19).

Employing the notation

$$S_{xi}(\eta) \equiv - \sum_{v=0}^{\infty} \eta(q^v x) \frac{U_i(x)}{U_i(q^{v+1} x)},$$

and using for each system that particular formal solution which is obtained by adding the solution $u_i(x)$ of the homogeneous equation to the solution derived from the first equation (19), we have

$$f_i^{(1)}(x) = u_i(x),$$

$$f_i^{(2)}(x) = u_i(x) + S_{xi} \left(\sum_{j=1}^n \psi_{ij} u_j \right),$$

$$f_i^{(3)}(x) = u_i(x) + S_{xi} \left\{ \sum_{j=1}^n \psi_{ij} u_j + \sum_{j=1}^n \psi_{ij} S_{xj} \left(\sum_{k=1}^n \psi_{jk} u_k \right) \right\},$$

.....

By actual substitution one may readily verify that a solution of system (14) is obtained in the limit functions (if they exist) of the sequences $f_i^{(1)}, f_i^{(2)}, \dots$, $i = 1, \dots, n$; that is, in

$$\begin{aligned} f_i(x) = & u_i(x) + S_{xi} \left(\sum_{j=1}^n \psi_{ij} u_j \right) + S_{xi} \left\{ \sum_{j=1}^n \psi_{ij} S_{xj} \left(\sum_{k=1}^n \psi_{jk} u_k \right) \right\} \\ & + S_{xi} \left[\sum_{j=1}^n \psi_{ij} S_{xj} \left\{ \sum_{k=1}^n \psi_{jk} S_{xk} \left(\sum_{l=1}^n \psi_{kl} u_l \right) \right\} \right] + \dots \quad (i = 1, \dots, n). \end{aligned} \quad (20)$$

For a fixed j let us take $C_i(x) = \delta_{ij}$. We thus obtain a particular formal solution of (14), which we will denote by $f_{ij}(x)$, $i = 1, \dots, n$. Making this substitution for every j , we obtain the n particular formal solutions

$$f_{1j}(x), f_{2j}(x), \dots, f_{nj}(x) \quad (j = 1, \dots, n),$$

where

$$\begin{aligned} f_{ij}(x) = & \delta_{ij} U_j(x) + S_{xi} (\psi_{ij} U_j) + S_{xi} \left\{ \sum_{k=1}^n \psi_{ik} S_{xk} (\psi_{kj} U_j) \right\} \\ & + S_{xi} \left[\sum_{k=1}^n \psi_{ik} S_{xk} \left\{ \sum_{l=1}^n \psi_{kl} S_{xl} (\psi_{lj} U_j) \right\} \right] + \dots \quad (i, j = 1, \dots, n). \end{aligned} \quad (21)$$

There exist constants R' , M , c_1 and c_2 such that, for every i and j ,

$$|\psi_{ij}(x)| < M|x|^{-s-1}, \quad c_1 < |U_j(x)x^{-\mu_j}| < c_2, \quad |x| \geq R'.$$

Then the convergence of all the series in (21) and of the series (21) itself for $|x| \geq R'$ will follow readily from similar convergence in the series

$$\bar{S}_{xi} \left(\frac{Mc_2}{x^{s+1-\mu_j}} \right) + \bar{S}_{xi} \left\{ \sum_{k=1}^n \frac{M}{x^{s+1}} \bar{S}_{xk} \left(\frac{Mc_2}{x^{s+1-\mu_j}} \right) \right\} + \dots, \quad (22)$$

where

$$\bar{S}_{xi}(\gamma) \equiv \sum_{v=0}^{\infty} |\gamma(q^v x)| \frac{|U_i(x)|}{|U_i(q^{v+1}x)|}.$$

We will suppose now that the arbitrary integer s has been chosen greater than twice the absolute value of any μ . Considering the first term of (22), we have

$$\bar{S}_{xi} \left(\frac{Mc_2}{x^{s+1-\mu_j}} \right) < \sum_{v=0}^{\infty} \left| \frac{Mc_2}{(q^v x)^{s+1-\mu_j}} \frac{c_2}{c_1} \right| = |x^{\mu_j-s-1}| \sum_{v=0}^{\infty} \frac{Mc_2^2}{|c_1 q^{v(s+1-\mu_j)}|} = q_1 |x^{\mu_j-s-1}|,$$

where q_1 denotes the sum of the last written series—a series which converges, since $s > |\mu_j|$. From this result it follows that the series $x^{-\mu_j} S_{xi}(\psi_{ij} U_j)$ is uniformly convergent for $|x| \geq R'$ and that its sum is less in absolute value than $q_1 |x^{-s-1}|$. Moreover, each term of the series is analytic for $|x| \geq R'$; and, since

the series converges uniformly throughout the region, the sum is analytic in the same region.

Taking the second term of (22) and using the result of the last paragraph, we have

$$\begin{aligned}\bar{S}_{xi} \left\{ \sum_{k=1}^n \frac{M}{x^{s+1}} \bar{S}_{xk} \left(\frac{Mc_2}{x^{s+1-\mu_j}} \right) \right\} &< \bar{S}_{xi} \left\{ \frac{Mn}{x^{s+1}} q_1 x^{\mu_j-s-1} \right\} = \sum_{v=0}^{\infty} \frac{Mnq_1}{|(q^v x)^{2s+2-\mu_j}|} \frac{c_2}{c_1} \\ &= q_1 |x^{\mu_j-2s-2}| \sum_{v=0}^{\infty} \frac{Mnc_2}{c_1 |q^{v(2s+2-\mu_j)}|} = q_1 q_2 |x^{\mu_j-2s-2}|,\end{aligned}$$

where q_2 denotes the sum of the last preceding series. From this we conclude that the third term in the series in (21) is in absolute value less than $q_1 q_2 |x^{\mu_j-2s-2}|$, and that it is the product of x^{μ_j} by a function which is analytic for $|x| \geq R'$.

Continuing this process we reach the result that every term in (21) is the product of x^{μ_j} by a function which is analytic for $|x| \geq R'$, and that the series is term by term less in absolute value than the series

$$\delta_{ij} c_2 |x^{\mu_j}| + q_1 |x^{\mu_j-s-1}| + q_1 q_2 |x^{\mu_j-2s-2}| + q_1 q_2 q_3 |x^{\mu_j-3s-3}| + \dots, \quad (23)$$

where

$$q_1 = \sum_{v=0}^{\infty} \frac{Mc_2^2}{c_1 |q^{v(s+1-\mu_j)}|}, \quad q_a = \sum_{v=0}^{\infty} \frac{Mnc_2}{c_1 |q^{v(as+a-\mu_j)}|}, \quad a = 2, 3, \dots.$$

Since q_2, q_3, \dots is a decreasing sequence with the limit zero, it follows that series (23) converges when $|x| > 1$. Hence there exists an R'' , $R'' > 1$, $R'' \geq R'$, such that the series obtained by multiplying (21) term by term by $x^{-\mu_j}$ is uniformly convergent and its terms are analytic for $|x| \geq R''$. Hence all the functions $x^{-\mu_j} f_{ij}(x)$, $i, j = 1, \dots, n$, defined by (21) are analytic for $|x| \geq R''$.

Moreover, since the series in (21) is term by term less than the series (23), and since the limit as x approaches infinity of $U_j(x)x^{-\mu_j}$ is 1, it follows that

$$\lim_{x=\infty} x^{-\mu_i} f_{ij}(x) = \delta_{ij} \quad (i, j = 1, \dots, n).$$

From this we see that the determinant of which the elements are $f_{ij}(x)$ is not identically zero, and this is clearly a necessary and sufficient condition that the solutions $f_{ij}(x)$ are linearly independent in the sense that no solution is expressible linearly in terms of the others, the multipliers being functions satisfying the equation $C(qx) = C(x)$.

If now we write

$$G_{ij}^{(\infty)}(x) = e^{\frac{\alpha}{2} \log q (\eta^2 + \eta)} \sum_{k=1}^n \pi_{ik} \left(\frac{1}{x} \right) f_{kj}(x), \quad \eta = \log x / \log q, \quad (i, j = 1, \dots, n),$$

it is clear from (11) that we have n solutions

$$G_{1j}^{(\infty)}(x), G_{2j}^{(\infty)}(x), \dots, G_{nj}^{(\infty)}(x) \quad (j = 1, \dots, n)$$

of equation (1). Furthermore, since the transformation (11) is non-singular, it is evident that these solutions are linearly independent and constitute therefore a fundamental system of solutions of (1). On account of the properties of $f_{ij}(x)$ it is clear that we may write

$$G_{ij}^{(\infty)}(x) = e^{\frac{\alpha}{2} \log q (\eta^2 + \eta)} x^\mu V_{ij}^{(\infty)}(x) \quad (i, j = 1, \dots, n),$$

where $V_{ij}^{(\infty)}(x)$ is analytic in the neighborhood of infinity.

If an expression of the above form for $G_{ij}^{(\infty)}(x)$ is substituted in (1), $V_{ij}^{(\infty)}(x)$ being expanded as a power-series in $1/x$, and if the constants are determined by direct reckoning, it will turn out that each of the n formal solutions obtained in this way is unique up to a constant multiplier. Consequently, if these constant multipliers are properly chosen, these formal solutions are the actual solutions whose existence has just been determined.

This solution of (1) has been obtained by starting from the normal form (14) and employing the method of successive approximation. Starting from (15) and working in a similar way, it may be shown that we have also a fundamental system of solutions

$$G_{1j}^{(0)}(x), G_{2j}^{(0)}(x), \dots, G_{nj}^{(0)}(x) \quad (j = 1, \dots, n)$$

which may be written in the form

$$G_{ij}^{(0)}(x) = e^{\frac{\beta}{2} \log q (\eta^2 + \eta)} x^{m_j} V_{ij}^{(0)}(x) \quad (i, j = 1, \dots, n),$$

where $V_{ij}^{(0)}(x)$ is analytic in the neighborhood of infinity. and that this solution may be obtained by substituting in the equation as in the foregoing case and determining the constants in a direct way. The principal modification in the argument consists in using the second instead of the first formal solution (19) of the non-homogeneous equation (18). The discussion follows so closely along the lines of the preceding that it is unnecessary to repeat it.

The argument above has been carried out on the assumption that $|q| > 1$. If $|q| < 1$, the only essential modification of the argument consists in interchanging the rôles of the two formal solutions (19) of (18). In this case the

second should be used for solutions about infinity and the first for solutions about zero. The results as to the nature of the solutions at these points still remain valid.

The nature of each of the functions of the two fundamental systems of solutions obtained above is so far determined only in the neighborhood of infinity and of zero respectively. In order to determine their nature outside of these regions, we work as follows: If we consider the first two members of (1) and solve for $G_i(x)$ in terms of $G_i(qx)$, $i = 1, \dots, n$, we may write the result in the form

$$G_i(x) = q^{-a} x^{-a} \sum_{j=1}^n \bar{A}_{ij}(qx) G_j(qx) \quad (i = 1, \dots, n). \quad (24)$$

Then from (1) itself and the preceding system, we have the relations

$$G_{ij}(qx) = q^a x^a \sum_{k=1}^n A_{ik}(x) G_{kj}(x) \quad (i, j = 1, \dots, n), \quad (25)$$

$$G_{ij}(x) = q^{-a} x^{-a} \sum_{k=1}^n \bar{A}_{ik}(qx) G_{kj}(qx) \quad (i, j = 1, \dots, n), \quad (26)$$

where $G_{ij}(x)$ is to be identified with either $G_{ij}^{(\infty)}(x)$ or $G_{ij}^{(0)}(x)$.

Let r_1, r_2, r_3, \dots be an infinite sequence of increasing numbers such that $r_{i+1} = |q| r_i$ or $r_{i+1} = |q|^{-1} r_i$, $i = 1, 2, \dots$, according as $|q| > 1$ or $|q| < 1$. Consider the infinite system of circles of radii r_1, r_2, r_3, \dots and center at the origin, and assume that r_1 has been so chosen that the first two of these circles lie within the region in which $V_{ij}^{(0)}(x)$ is analytic. These circles divide the part of the plane external to the first one into an infinite system of circular rings, which we shall call *fundamental regions*. If we know the value of $G_{ij}^{(0)}(x)$ at every point in one of these rings, we can compute its value at any point in the next larger by means of (25) if $|q| > 1$, and by means of (26) if $|q| < 1$. From this fact it is easy to determine the position of the singularities of $G_{ij}^{(0)}(x)$. If $\mu = q^t \nu$, where t is any positive or negative integer (not zero), we shall say that μ is *externally* or *internally congruent* to ν according as $|\mu| > |\nu|$ or $|\mu| < |\nu|$. Excluding the points zero and infinity, we see that the remaining singularities of each function $G_{ij}^{(0)}(x)$ are all included in the set of points externally congruent to the singularities of $A_{ik}(x)$ or of $\bar{A}_{ik}(x)$, $i, k = 1, \dots, n$, according as $|q| > 1$ or $|q| < 1$.

In a similar way it is readily shown that the singularities of each function $G_{ij}^{(\infty)}(x)$, other than zero and infinity, are all included in the set of points inter-

nally congruent to the singularities of $\bar{A}_{ik}(x)$ or of $A_{ik}(x)$, $i, k = 1, \dots, n$, according as $|q| > 1$ or $|q| < 1$.

Furthermore, if $A_{ik}(x)$, $i, k = 1, \dots, n$, are rational functions, it is clear that the singularities (other than zero and infinity) of the functions in either solution consist entirely of poles.

The principal results which we have obtained may be stated in the following theorem:

THEOREM I. *The system of q -difference equations (1) or (24) has two fundamental systems of solutions of the forms*

$$G_{ij}^{(\infty)}(x) = e^{\frac{\alpha}{2} \log q (\eta^2 + \eta)} x^{\mu_j} V_{ij}^{(\infty)}(x), \quad \eta = \frac{\log x}{\log q}, \quad (i, j = 1, \dots, n),$$

$$G_{ij}^{(0)}(x) = e^{\frac{\beta}{2} \log q (\eta^2 + \eta)} x^{m_j} V_{ij}^{(0)}(x) \quad (i, j = 1, \dots, n),$$

where $V_{ij}^{(\infty)}(x)$ and $V_{ij}^{(0)}(x)$ are analytic in the neighborhood of infinity and of zero respectively. These solutions may be determined by substitution (a power-series being assumed for each of the functions V) and direct reckoning out of constants. Besides the points zero and infinity, all the singularities of $G_{ij}^{(\infty)}(x)$ are included in the set of points internally congruent to the singularities of $\bar{A}_{ik}(x)$ or of $A_{ik}(x)$, $i, k = 1, \dots, n$, according as $|q| > 1$ or $|q| < 1$; and all the singularities of $G_{ij}^{(0)}(x)$ are included in the set of points externally congruent to the singularities of $A_{ik}(x)$ or of $\bar{A}_{ik}(x)$, $i, k = 1, \dots, n$, according as $|q| > 1$ or $|q| < 1$. The general solution $G_i(x)$, $i = 1, \dots, n$, may be written in either of the forms

$$G_i(x) = \sum_{j=1}^n C_j^{(\infty)}(x) G_{ij}^{(\infty)}(x) = \sum_{j=1}^n C_j^{(0)}(x) G_{ij}^{(0)}(x) \quad (i = 1, \dots, n),$$

where the functions $C_j^{(\infty)}(x)$ and $C_j^{(0)}(x)$, $j = 1, \dots, n$, satisfy the equation $C(qx) = C(x)$ but are otherwise arbitrary.

Further, if $A_{ik}(x)$, $i, k = 1, \dots, n$, are rational functions of x , the singularities (other than zero and infinity) of the functions in each of the two fundamental solutions consist entirely of poles.

Let us now consider a single q -difference equation of the n -th order in the two forms

$$\begin{aligned} H(q^n x) + (q^n x)^\alpha A_1(x) H(q^{n-1} x) + (q^n x)^\alpha (q^{n-1} x)^\alpha A_2(x) H(q^{n-2} x) \\ + \dots + (q^n x)^\alpha \dots (q^2 x)^\alpha (qx)^\alpha A_n(x) H(x) = 0, \end{aligned} \quad (27)$$

and

$$\begin{aligned} H(q^n x) + (q^n x)^\beta B_1(x) H(q^{n-1} x) + (q^n x)^\beta (q^{n-1} x)^\beta B_2(x) H(q^{n-2} x) \\ + \dots + (q^n x)^\beta \dots (q^2 x)^\beta (qx)^\beta B_n(x) H(x) = 0, \end{aligned} \quad (28)$$

where the unknown quantities entering into the equation are defined as follows:

1. α, β, q are constants and $|q| \neq 1$.
2. The functions $A_i(x)$ and $B_i(x)$ are single-valued and

$$A_i(x) = A_i + A'_i x^{-1} + A''_i x^{-2} + \dots \quad (i = 1, \dots, n) \quad |x| \geq R,$$

$$B_i(x) = B_i + B'_i x + B''_i x^2 + \dots \quad (i = 1, \dots, n) \quad |x| \leq r.$$

3. The constants A_i and the constants B_i are such that the characteristic equations at infinity and at zero,

$$\rho^n + A_1 \rho^{n-1} + A_2 \rho^{n-2} + \dots + A_{n-1} \rho + A_n = 0, \quad (29)$$

$$\rho^n + B_1 \rho^{n-1} + B_2 \rho^{n-2} + \dots + B_{n-1} \rho + B_n = 0, \quad (30)$$

have their respective roots a_1, \dots, a_n and b_1, \dots, b_n of simple character.

The transformations

$$H(x) = e^{\frac{\alpha}{2} \log q (\eta^2 + \eta)} \bar{H}(x), \quad \eta = \log x / \log q,$$

$$H(x) = e^{\frac{\beta}{2} \log q (\eta^2 + \eta)} \bar{H}'(x),$$

carry equations (27) and (28) respectively over into

$$\bar{H}(q^n x) + A_1(x) \bar{H}(q^{n-1} x) + \dots + A_n(x) \bar{H}(x) = 0, \quad (31)$$

$$\bar{H}'(q^n x) + B_1(x) \bar{H}'(q^{n-1} x) + \dots + B_n(x) \bar{H}'(x) = 0. \quad (32)$$

It is evident that these two equations are equivalent respectively to the two systems

$$\bar{H}_1(qx) = \sum_{j=1}^n \{-A_j(x)\} \bar{H}_j(x), \quad \bar{H}_i(qx) = \bar{H}_{i-1}(x) \quad (i = 2, \dots, n), \quad (33)$$

$$\bar{H}'_1(qx) = \sum_{j=1}^n \{-B_j(x)\} \bar{H}'_j(x), \quad \bar{H}'_i(qx) = \bar{H}'_{i-1}(x) \quad (i = 2, \dots, n), \quad (34)$$

where $\bar{H}_n(x) = \bar{H}(x)$ and $\bar{H}'_n(x) = \bar{H}'(x)$. The characteristic equation of (33) at infinity takes the form (29), and the characteristic equation of (34) at zero takes the form (30).

Equations (33) and (34) are of the form (3) and (4) respectively and consequently the preceding discussion is applicable to the former systems. Corresponding to the solved form (24) of (1) are analogous solved forms of (31) and (33) respectively; namely,

$$\bar{H}(x) + \bar{A}_1(x) \bar{H}(qx) + \bar{A}_2(x) \bar{H}(q^2 x) + \dots + \bar{A}_n(x) \bar{H}(q^n x) = 0, \quad (35)$$

$$\bar{H}_i(x) = \bar{H}_{i+1}(qx), \quad (i = 1, \dots, n-1); \quad \bar{H}_n(x) = \sum_{j=1}^n \{-\bar{A}_{n-j+1}(x)\} \bar{H}_j(qx), \quad (36)$$

where the functions $\bar{A}_i(x)$, $i = 1, \dots, n$, are readily expressed in terms of the functions $A_i(x)$, $i = 1, \dots, n$.

If now we apply our previous results to the systems of equations and interpret them with reference to the single equation of the n -th order, we obtain the following theorem:

THEOREM II. *The n -th order q -difference equation (27), (28) or (35) has two fundamental systems of solutions of the forms*

$$H_i^{(\infty)}(x) = e^{\frac{\alpha}{2} \log q (\eta^2 + \eta)} x^{\mu_i} V_i^{(\infty)}(x), \quad \eta = \frac{\log x}{\log q}, \quad (i = 1, \dots, n),$$

$$H_i^{(0)}(x) = e^{\frac{\beta}{2} \log q (\eta^2 + \eta)} x^{m_i} V_i^{(0)}(x) \quad (i = 1, \dots, n),$$

where $V_i^{(\infty)}(x)$ and $V_i^{(0)}(x)$ are analytic in the neighborhood of infinity and of zero respectively. These solutions may be determined by substitution (a power-series being assumed for each of the functions V) and direct reckoning out of constants. Besides the points zero and infinity, all the singularities of $H^{(\infty)}(x)$ are included in the set of points internally congruent to the singularities of $\bar{A}_i(x)$ or of $A_i(x)$, $i = 1, \dots, n$, according as $|q| > 1$ or $|q| < 1$; and all the singularities of $H_i^{(0)}(x)$ are included in the set of points externally congruent to the singularities of $A_i(x)$ or of $\bar{A}_i(x)$, $i = 1, \dots, n$, according as $|q| > 1$ or $|q| < 1$. The general solution $H(x)$ may be written in either of the forms

$$H(x) = \sum_{j=1}^n C_j^{(\infty)}(x) H_j^{(\infty)}(x) = \sum_{j=1}^n C_j^{(0)}(x) H_j^{(0)}(x),$$

where the functions $C_j^{(\infty)}(x)$ and $C_j^{(0)}(x)$, $j = 1, \dots, n$, satisfy the equation $C(qx) = C(x)$ but are otherwise arbitrary.

Further, if $A_i(x)$, $i = 1, \dots, n$, are rational functions of x , the singularities (other than zero and infinity) of the functions in each of the two fundamental solutions consist entirely of poles.

REMARK. In view of the preceding work it is obvious that the results of this section may be obtained by direct reckoning out of the solutions, by the power-series method suggested above, together with appropriate convergence proofs. Such a method, however, is not available in the case of difference equations. Consequently it has seemed best to employ the method of successive approximation (already used by me for difference equations), so as to develop the two parts of this general theory along the same lines and by the same methods. The matrix method used by Birkhoff for difference equations is also applicable to q -difference equations; it furnishes therefore a second single method by which both cases may be treated.

§ 2. *Relations between the two Fundamental Solutions.*

The question of the relations existing between the two fundamental solutions of system (1) furnishes an interesting problem. It is clear that either of these solutions may be expressed in terms of the other. Thus we have

$$G_{ij}^{(0)}(x) = \sum_{k=1}^n C_{ik}(x) G_{kj}^{(\infty)}(x) \quad (i, j = 1, \dots, n), \quad (37)$$

$$G_{ij}^{(\infty)}(x) = \sum_{k=1}^n \bar{C}_{ik}(x) G_{kj}^{(0)}(x) \quad (i, j = 1, \dots, n), \quad (38)$$

where $C_{ik}(x)$ and $\bar{C}_{ik}(x)$ are functions satisfying the equation $C(qx) = C(x)$. Our problem reduces, then, to that of the determination of the nature of these functions $C_{ik}(x)$ and $\bar{C}_{ik}(x)$. In the general case they are of a complicated character. But in the most interesting and important case, namely, that in which the coefficients of the original equation are polynomials (or even when they are merely rational functions), the result takes a simple and elegant form.

We suppose that system (1), subject to the previously assigned conditions, is also such that it may be written in the form

$$G_i(qx) = \sum_{k=1}^n P_{ik}(x) G_k(x) \quad (i = 1, \dots, n), \quad (39)$$

where the coefficients $P_{ik}(x)$ are polynomials in x of degree α or less. The constant β which enters into system (1) is in this case zero.

According as $|q| > 1$ or $|q| < 1$, the functions $G_{ij}^{(0)}(x)$ or $G_{ij}^{(\infty)}(x)$ are analytic except at zero and infinity. We will consider first the case in which $|q| > 1$. If (37) is solved for $C_{ik}(x)$, $i, k = 1, \dots, n$, in terms of $G_{ij}^{(0)}(x)$ and $G_{ij}^{(\infty)}(x)$, it is readily seen that every function $C_{ik}(x)$ is analytic except perhaps at zero, infinity, the singularities of $G_{ij}^{(\infty)}(x)$, and the zeros of the function defined by the determinant $|G_{ij}^{(\infty)}(x)|$. But on account of the properties of the functions $G_{ij}^{(\infty)}(x)$ and of the form of the determinant (see Theorem I), it is clear that there exists a fundamental region (sufficiently remote from the origin) in which the functions $G_{ij}^{(\infty)}(x)$, $i, j = 1, \dots, n$, have no singularity and the function $|G_{ij}^{(\infty)}(x)|$ has no zero. Hence the functions $C_{ik}(x)$, $i, k = 1, \dots, n$, are analytic in this fundamental region; and therefore they are analytic throughout the plane, except at zero and infinity.

In case $|q| < 1$, we may argue in a similar way to show that the functions $\bar{C}_{ik}(x)$, $i, k = 1, \dots, n$, are analytic throughout the plane, except at zero and infinity.

It is clear that if the coefficients $P_{ik}(x)$ in (39) are restricted to be rational (but not necessarily polynomial), a discussion analogous to that in the foregoing case can be readily carried out. If $|q| > 1$, the functions $C_{ik}(x)$, away from zero and infinity, will be analytic except for poles; and if $|q| < 1$, the functions $\bar{C}_{ik}(x)$ will have this property.

These results may be stated in the following theorem:

THEOREM III. *Let a system of q-difference equations (1) be written in the form*

$$G_i(qx) = \sum_{k=1}^n P_{ik}(x) G_k(x) \quad (i = 1, \dots, n);$$

and let the two simple fundamental solutions be connected by the relations

$$G_{ij}^{(r)}(x) = \sum_{k=1}^n C_{ik}(x) G_{kj}^{(s)}(x), \quad C_{ik}(qx) = C_{ik}(x) \quad (i, j, k = 1, \dots, n),$$

where $r, s = 0, \infty$ or $\infty, 0$ according as $|q| > 1$ or $|q| < 1$. Then:

If the coefficients $P_{ik}(x)$ are polynomials in x , the functions $C_{ik}(x)$ are analytic except at zero and infinity;

If the coefficients $P_{ik}(x)$ are rational in x , the functions $C_{ik}(x)$, away from zero and infinity, are analytic except for poles.

An inverse to some of the preceding results may be stated in the form of the following theorem:

THEOREM IV. *Let $G_{ij}^{(0)}(x)$ and $G_{ij}^{(\infty)}(x)$, $i, j = 1, \dots, n$, be two sets of n^2 single-valued functions which, away from zero and infinity, are analytic except for poles, and which have the property*

$$\lim_{x \rightarrow 0} G_{ij}^{(0)}(x) e^{\frac{-\beta}{2} \log q (\eta^2 + \eta)} x^{-m_i} = V_{ij}^{(0)}, \quad |V_{ij}^{(0)}| \neq 0; \quad m_i - m_j \neq \text{integer}, i \neq j,$$

$$\lim_{x \rightarrow \infty} G_{ij}^{(\infty)}(x) e^{\frac{-\alpha}{2} \log q (\eta^2 + \eta)} x^{-\mu_i} = V_{ij}^{(\infty)}, \quad |V_{ij}^{(\infty)}| \neq 0; \quad \mu_i - \mu_j \neq \text{integer}, i \neq j,$$

where α and β are integers (positive, negative or zero), $\eta = \log x / \log q$, the two sets of functions being connected by the relations

$$G_{ij}^{(0)}(x) = \sum_{k=1}^n C_{ik}(x) G_{kj}^{(\infty)}(x), \quad C_{ik}(qx) = C_{ik}(x), \quad (i, j, k = 1, \dots, n).$$

Then the two sets of functions $G_{ij}^{(0)}(x)$ and $G_{ij}^{(\infty)}(x)$ are the two simple fundamental solutions of a system of q-difference equations

$$G_i(qx) = \sum_{k=1}^n P_{ik}(x) G_k(x) \quad (i = 1, \dots, n),$$

in which the coefficients $P_{ik}(x)$ are rational in x .

The proof of the theorem requires the determination of the existence and the properties of functions $P_{ik}(x)$ such that

$$G_{ij}^{(0)}(qx) = \sum_{k=1}^n P_{ik}(x) G_{kj}^{(0)}(x) \quad (i, j = 1, \dots, n), \quad (40)$$

$$G_{ij}^{(\infty)}(qx) = \sum_{k=1}^n P_{ik}(x) G_{kj}^{(\infty)}(x) \quad (i, j = 1, \dots, n). \quad (41)$$

Each of the two systems of equations may be solved for $P_{ik}(x)$. The two representations of $P_{ik}(x)$ must be consistent, in view of the relations connecting $G_{ij}^{(0)}(x)$ and $G_{ij}^{(\infty)}(x)$. From either of them it is clear that the functions $P_{ik}(x)$, away from zero and infinity, are single-valued and analytic except for poles.

If (40) is solved for $P_{ik}(x)$ in terms of $G_{ij}^{(0)}(x)$ and $G_{ij}^{(0)}(qx)$, $i, j, k = 1, \dots, n$, and the limit for $x=0$ is taken, then, in view of the first limit in the hypothesis of the theorem, we have

$$\lim_{x=0} P_{ik}(x) x^{-\beta} = \frac{|V_{rs}^{(ik)}|}{|V_{rs}^{(0)}|},$$

where $|V_{rs}^{(0)}|$ is the determinant with $V_{rs}^{(0)}$ in the r -th row and s -th column and $|V_{rs}^{(ik)}|$ is what $|V_{rs}^{(0)}|$ becomes when the k -th row is replaced by $q^m V_{ij}^{(0)}$, $j=1, \dots, n$. From this it follows that $P_{ik}(x)$ has at $x=0$ a zero of order β (or a pole of order $-\beta$ if β is negative). Furthermore, the q -difference equation has at zero the characteristic equation

$$|P_{ik} - \delta_{ik}\rho| = 0, \quad P_{ik} = \frac{|V_{rs}^{(ik)}|}{|V_{rs}^{(0)}|}.$$

A direct reckoning will show that the roots of this equation are $\rho = q^m$, $j=1, \dots, n$.

In like manner we may employ (41) and the second limit in the hypothesis of the theorem to show that $P_{ik}(x)$ has at infinity a pole of order α (or a zero of order $-\alpha$ if α is negative) and that the roots of the characteristic equation at infinity are $\rho = q^n$, $j=1, \dots, n$.

From the fact that $P_{ik}(x)$ is analytic except for poles we conclude that it is rational in x . From the characteristic equations at zero and infinity respectively, we conclude that the q -difference equation of the theorem has solutions of the requisite property at each of these points. This completes the demonstration of the theorem as stated.

§ 3. The Case when $|q| = 1$.

When $|q| = 1$, the theory of the q -difference equation is essentially different from that when $|q| \neq 1$, previously considered. In treating the former we distinguish two cases, according as there is or is not a positive integer n such that $q^n = 1$.

CASE I. We will suppose first that there exists no integer n such that $q^n = 1$. That there is in this case in general no analytic solution (not identically zero) is easily seen from the consideration of a first-order equation with constant coefficients:

$$f(qx) = af(x), \quad a > 1.$$

We have evidently

$$f(q^a x) = af(q^{a-1}x) = a^2f(q^{a-2}x) = \dots = a^af(x).$$

For any x it is clear that there exists an α greater than any preassigned M and such that $q^\alpha x$ is less than any preassigned distance ϵ from x . When $f(x)$ is not equal to zero at the given point x the quotient of the functional values at $q^\alpha x$ and x , that is $f(q^\alpha x)/f(x)$, can therefore be made as large as one pleases, although $q^\alpha x$ and x are separated by a less distance than ϵ . Hence the equation $f(qx) = af(x)$ has no continuous solution except $f(x) \equiv 0$.

CASE II. We will suppose now that q is a root of unity and that n is the least integer such that $q^n = 1$. In the present case it is more convenient to consider from the outset a single equation than a system of equations, for evidently any single equation of any order can be written in the form

$$\psi_1(x)f(qx) + \psi_2(x)f(q^2x) + \dots + \psi_n(x)f(q^nx) = 0. \quad (42)$$

That solutions (not identically zero) do not exist when the ψ 's are unrestricted is readily shown; for we have the following system of equations:

$$\left. \begin{aligned} & \psi_1(x)f(qx) + \psi_2(x)f(q^2x) + \dots + \psi_n(x)f(q^n x) = 0, \\ & \psi_n(qx)f(qx) + \psi_1(qx)f(q^2x) + \dots + \psi_{n-1}(qx)f(q^n x) = 0, \\ & \psi_{n-1}(q^2x)f(qx) + \psi_n(q^2x)f(q^2x) + \dots + \psi_{n-2}(q^2x)f(q^n x) = 0, \\ & \dots\dots\dots, \\ & \psi_2(q^{n-1}x)f(qx) + \psi_3(q^{n-1}x)f(q^2x) + \dots + \psi_1(q^{n-1}x)f(q^n x) = 0. \end{aligned} \right\} \quad (43)$$

This system is consistent only if its determinant is zero; that is, if

$$\begin{vmatrix} \psi_1(x) & \psi_2(x) & \dots & \psi_n(x) \\ \psi_n(qx) & \psi_1(qx) & \dots & \psi_{n-1}(qx) \\ \psi_{n-1}(q^2x) & \psi_n(q^2x) & \dots & \psi_{n-2}(q^2x) \\ \dots & \dots & \dots & \dots \\ \psi_2(q^{n-1}x) & \psi_3(q^{n-1}x) & \dots & \psi_1(q^{n-1}x) \end{vmatrix} = 0. \quad (44)$$

We therefore suppose that the coefficients $\psi_i(x)$ are connected by the relation (44).

So far the only restriction on q is that it is a primitive n -th root of unity. The numbers $q, q^2, q^3, \dots, q^{n-1}, q^n$ are all the n -th roots of unity, and hence this set of powers contains the primitive n -th root of unity of smallest argument. Without loss of generality we may suppose that this root is q itself; and this we shall do. Then the points q, q^2, \dots, q^n are situated on the unit circle at equal intervals, in such wise that in making a positive circuit along the circumference the points are encountered in the order written. If rays are drawn from the origin through these n points, the plane is divided into n equal sectors; and these we shall call a set of fundamental regions of the plane. If these rays are all rotated about the origin through the same angle, then in their new position they will define another set of fundamental regions.

We have seen that a necessary condition for the existence of a solution (not identically zero) is that the determinant in (44) shall be of rank $n-1$ at most. We suppose in general that this determinant is of rank $n-\alpha$, where α is a number of the sequence 1, 2, \dots , $n-1$. Then, among the quantities

$$f(qx), f(q^2x), \dots, f(q^nx),$$

there is a set α in number in terms of which every other can be expressed linearly. Hence there are α fundamental regions in which $f(x)$ may be assigned at will; it is then determined throughout the entire plane. On account of this fact we shall say that the order of the equation is α .

When the equation is of order 1, in accordance with this definition, there is a single relation among the coefficients; otherwise there is more than one. The general case is therefore that in which the order is 1. Since in this case a solution is determined when its value is assigned in a single fundamental region, it is clear that the most general solution is of the form $C(x)f_1(x)$, where $f_1(x)$ is any particular solution (not identically zero) and $C(x)$ is an arbitrary function satisfying the equation $C(qx) = C(x)$.

In the case of a first-order equation it is easy to obtain a particular solution in explicit finite form. The ordinary theory of linear algebraic equations applied to the system (43) yields us the values of the ratios

$$f(qx) : f(q^2x) : \dots : f(q^n x);$$

and hence we have an equation of the form

$$f(qx) = \psi(x) f(x), \quad (45)$$

where $f(x)$ is the same as the $f(x)$ in (42). To find a solution of this equation, we proceed thus:

The product

$$f(q^n x) f(q^{n-1} x) \dots f(qx)$$

is unchanged by the substitution of qx for x ; and hence its value is a function satisfying the relation $C(qx) = C(x)$. Consequently there exists a particular solution $f_1(x)$ for which

$$f_1(q^n x) f_1(q^{n-1} x) \dots f_1(qx) = 1.$$

But

$$f_1(q^{i+1} x) = \psi(q^i x) f_1(q^i x).$$

Making repeated substitutions from this formula into the preceding equation, we have

$$\psi(q^{n-1} x) [\psi(q^{n-2} x)]^2 \dots [\psi(qx)]^{n-1} [\psi(x)]^n [f_1(x)]^n = 1.$$

Hence

$$f_1(x) = \{\psi(q^{n-1} x) [\psi(q^{n-2} x)]^2 \dots [\psi(qx)]^{n-1} [\psi(x)]^n\}^{-1/n}$$

is a particular solution of (45) and hence of (43); the general solution is $C(x)f_1(x)$, where $C(x)$ satisfies the equation $C(qx) = C(x)$ but is otherwise arbitrary.

The above discussion suggests a general method for solving (42) when the rank of its determinant in (44) is $n - \alpha$. In a form most convenient for this case the method may be described as follows: To any $n - \alpha$ independent equations of the set (43) adjoin other equations subject to the following conditions: 1) The n equations of the resulting set are independent and consistent. 2) The left members of the α new equations are invariant under the substitution of qx for x . They are linear in $f(q^n x), \dots, f(x)$. 3) The second members of these α equations are arbitrary functions of x satisfying the relation $C(qx) = C(x)$.

One of the possible forms for these equations is

$$\lambda(q^{n-1}x)f(q^{n-1}x) + \dots + \lambda(qx)f(qx) + \lambda(x)f(x) = C(x),$$

where $\lambda(x)$ is any conveniently chosen function of x .

If the system is solved algebraically for $f(q^n x), \dots, f(x)$, the resulting expression for $f(x)$ will contain linearly α arbitrary functions which satisfy the equation $C(qx) = C(x)$. And hence this gives the general solution of (42), since it is of order α .

In carrying out this process it is important to secure that the arbitrary functions shall enter into the solution independently; that is, in such way that they can not be combined in the solution into fewer functions of the same type.

We shall illustrate the above method by a treatment of the special case in which $q = \omega =$ cube root of unity of least argument and $\alpha = 2$. It is easy to show that the general equation for this case may be written in the form

$$f(\omega^2 x) + \psi(x)\psi(\omega x)f(\omega x) + \psi(x)f(x) = 0, \quad \psi(x)\psi(\omega x)\psi(\omega^2 x) = 1.$$

In accordance with our general method, if $\psi(x) \not\equiv 1$,* we may adjoin the equations

$$\begin{aligned} f(\omega^2 x) + f(\omega x) + f(x) &= C_1(x), \\ \omega^2 x f(\omega^2 x) + \omega x f(\omega x) + f(x) &= C_2(x). \end{aligned}$$

The value of $f(x)$ is obtained algebraically from the last three equations.

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* If $\psi(x) \equiv 1$, a fundamental system of solutions is 1 and x .